Mathematical Typesetting

Mathematical and Scientific Typesetting Solutions from Microsoft

Edited by Ross Mills & John Hudson
with contributions by Richard Lawrence and Murray Sargent

Dedicated to Donald E. Knuth

Printed in Canada.

Foreword

Geraldine Wade, font development manager

The need for well-designed, well-executed mathematical typesetting is as vital today as it has been throughout the history of modern sciences. Indeed, quality fonts and optimal composition for math setting are currently even more crucial, due to the importance of mathematics and science in our everyday lives and the use of computers in the preparation of documents including mathematical expressions with intricate structures. Additionally, reading on screen is becoming as important as reading in print; so there's an added dimension to the work necessary to produce a font or set of fonts that will be both legible and beautiful in both mediums.

Bringing math and design needs together to make a fully functional product that typesets math well is a challenge and an opportunity that ultimately enriches both disciplines. Math symbols and expressions contain a rich visual vocabulary waiting to be explored by the typographer. The art and skill necessary to make a set of well designed typefaces that can represent the needs of a mathematician are an invaluable aid to quality mathematical typesetting and increased comprehension for the reader and student.

In this project we believe we have made a step along the path toward improving mathematical typography for our customers worldwide.

About this book

This book introduces the work of the team of software engineers and managers, mathematicians and scientists, and font developers who have introduced the highest qualities of mathematical typesetting to the latest generation of Microsoft® Office and other products. This work is placed in an historical perspective highlighting the long tradition of collaboration between mathematicians and typographers to express in written form mathematical concepts and formulae.

The book focuses on the Cambria™ Math font implementation for mathematical typesetting, rather than on the math layout engine software, and is intended as an introduction for mathematicians and scientists as users of the font, and for designers and font developers interested in understanding the general principles of Microsoft's approach to mathematical typesetting and the features of the Cambria Math font. The book does not provide detailed technical documentation, which will be made available in other formats. It is written as a general overview, in order to be accessible to those whose interest is in the quality of the results of mathematical typesetting, rather than the minutiae of how those results are achieved.

The book concludes with a specimen of mathematical formulae typeset in Microsoft Word 2007, using the new math layout functionality and the Cambria Math font.

Introduction

Mathematics is a language. Like other languages it has dialects and specialist vocabularies, and it evolves to express new ideas and to name new objects and concepts. It also has its own writing system that, like many of the world's writing systems, has borrowed symbols from the orthographies of other languages and also invented new symbols. The evolving nature of the language of mathematics, and its particular need to express complex ideas in concise ways, has resulted in an especially productive writing system. Mathematical authors not only write according to established orthographical conventions, but frequently invent new conventions to express original ideas. Crucially, not only the visual symbols have significance in the writing system, but also their relative size and spatial relationships. Vertical and horizontal arrangement, enclosure within other symbols—which may grow or shrink relative to their contents—, changes in size or weight: all these elements make the writing system of mathematics among the most dynamic and, for the typesetter, most challenging in the world.

The poet-typographer Robert Bringhurst calls writing "the solid form of language". Reducing the ideas of mathematicians and scientists to the very solid form of typeset text has been, and remains, a challenge for authors, editors and typographers. The result, seen on the printed page or, increasingly, on the computer screen may seem quite liberated and fluid compared to the written forms of most other languages: the symbols spill outward and upward across the page, dance around lines and braces, jump up from their baselines and nestle beside their companions. A page of well-set equations has a particular visual beauty that frequently seems more open and expansive than the stiff rows or columns of letters that characterize most writing systems. But this expansiveness and apparent fluidity is hard-won. The measure of control required to accurately size and position each element is very great, and the rendering of the mathematical notation in typeset form has long been a major challenge to typographers, printers and the developers of typesetting equipment. But as the historical section of this booklet reveals, there is also a natural affinity between the aims of mathematicians and those of typographers, in expressing as clearly as possible ideas and concepts. This has resulted in a long and fruitful collaboration between mathematicians, typographers and type designers.

Bringing together the skills and knowledge of mathematicians, researchers, computer scientists, programmers, text layout and font engineers, project managers, type designers and testers—acknowledged overleaf—was a major undertaking. The successful implementation of Microsoft's typesetting solutions is a testament to both individual and corporate commitment.

Acknowledgements

Math layout handler Eliyezer Kohen (initiator); Andrei Burago, Sergey Genkin, Victor Kozyrev, Igor Zverev, Alexander Vaschillo and Anton Sukhanov (developers); Vivek Garg, Mary Wang and Greg Heino (testers).

Cambria Math font Jelle Bosma, Robin Nicholas, Steve Matteson and Ross Mills (designers); Ross Mills, John Hudson, Mike Duggan and Andrei Burago (developers); Geraldine Wade and Greg Hitchcock (managers); Vivek Garg (tester).

Math font program library and editor Sergey Malkin (developer).

RichEdit Murray Sargent, Sasha Gil, Mikhail Baranovsky, Hon-Wah Chan and José Oglesby (developers); Zane Mumford (manager); Isao Yamauchi, Greg Heino, Yuriko Rosnow and Parag Palekar (testers).

Microsoft Word Ethan Bernstein, Said Abou-Hallawa, Ali Taleghani and Bryan Krische (developers); Jennifer Michelstein (manager); Jason Rajtar and Yi Zhang (testers).

Microsoft Math Calculator Jinsong Yu (developer); Ben Kunz (manager); Luke Kelly (tester).

Office Art Handwriting Microsoft Research developers, testers and project managers (Asia, Redmond, Belgrade)

Many thanks to Barbara Beeton (AMS), Asmus Freytag, Ron Whitney, Richard Lawrence, and Donald Knuth for their advice and very helpful discussions, and to Paula B. Croxon for editorial consultation on this book.

Historical perspectives

One of the goals of Microsoft's new mathematical typesetting solutions is to make high quality math setting available to users familiar with common word-processing programs, and to make it fast to learn and easy to get high quality results. Another goal is to incorporate the handling of complex mathematical typography into the existing architecture of OpenType® font technology, the RichEdit layout library and other common resources, so it can be utilized by a broad range of applications, not limited to the arrangement and printing of equations on paper. In this approach, Microsoft is building on its strengths and experience in text processing and font technology, but it is also building on centuries of work by mathematicians, scientists, educators, typographers and printers who have sought to give visual expression to mathematical concepts.

As the following overview shows, the history of mathematical typesetting is a history of distinguished collaboration, involving some of the greatest names in mathematics as well as many anonymous technicians, typesetters and printers.

Historical perspectives on the typography of mathematics

Richard Lawrence

The typesetting and printing of mathematics is today much easier than it has ever been. In the publishing field, preparation of press-ready material for the printer is now largely up to the author or editor, and can be produced on a personal computer. This comparative ease of preparation is complemented by the remarkable quality of the typesetting: both are, of course, related to the ingenuity and abilities of computer programmers, but they are also a testament to the skills of the type designers and typographers who prepare typefaces for the very exacting use they get in mathematics.

In the history of printing and typesetting, mathematics holds a peculiar place. It is both the material most feared by typesetting craftsmen and the inspiration for some of their greatest technical achievements. It is the subject matter that has perhaps inspired the most intense collaborations between authors and printers, yet it is often one of the least commercially rewarding for publishers. Through all this history, though, there has been an affinity of purpose between typographer and mathematician that has, at least in recent times, drawn the two together in overcoming common problems.

Good typography is transparent and does not place itself between the reader and comprehension: rather it silently aids understanding.

So a typeface fit to the subject matter does not distract from that matter, which is arranged into words and paragraphs so as to aid comprehension. The result is almost invariably an aesthetically pleasing, and often beautiful, marriage of form and function.

Mathematicians have a keen sense of beauty and the importance of utility in notation. An argument or idea clearly and concisely expressed will garner the description of "beautiful". But the concept of mathematical beauty is applied more widely than just to ideas. The expression of those ideas on paper may also be described as beautiful.

Written mathematics is an exceptional, perhaps unique, language:

> *… [*mathematicians*] have to express values, quantities and relationships by symbols which differ basically from those of the alphabet, in that they have no fixed phonetic value. … In the mathematician's world they still use sign language. … a soundless world in which the most complex and delicate statements can be made, to anyone in the world who can read them without any trouble about the language barrier.*
>
> Arthur Phillips, *The Monotype Recorder*, 40, 1956

Mathematical arguments have to be written in this *silent* language: it would be very taxing indeed to complete many mathematical discussions without pencil and paper or their equivalent. Given the subtle and perplexing nature of much mathematics, the details of the written form are clearly going to be important to the task of conveying the argument without introducing opacity or distraction. The aims of the mathematician are then the same as the typographer. A typographer and a mathematician examining the same piece of mathematics may not

"Typography may be defined as the craft of rightly disposing printing material in accordance with specific purpose; of so controlling the type as to aid to the maximum the reader's comprehension of the text."

Stanley Morison, *First principles of tpography,* 1951

"By relieving the brain of all unnecessary work, a good notation sets it free to concentrate on more advanced problems, and in effect increases the mental power of the human race."

Alfred North Whitehead, *An introduction to mathematics,* 1947 have an equal understanding of its meaning, but they are likely to have an equal appreciation of the aesthetics of its presentation: both seek to make it visually easy to follow and aesthetically pleasing.

The difficulty of mathematical typesetting

What used to make mathematics difficult for the typesetter and expensive for the publisher? One obvious answer is incomprehension of the subject matter. The difficulty of knowing how to arrange type is greater without any idea of what is being conveyed.

However, compositors (typesetters) with a bare minimum of schooling used to successfully typeset foreign languages in ignorance of their meaning. There are two substantial physical difficulties in setting mathematics. The first is the extraordinary diversity of individual characters or "sorts" in various styles (bold, italic, sans serif, etc.) and the many sizes required. Each sort in each style had to be literally at hand. Marshalling all the sorts required was a logistical problem and in the days of metal type, ensuring an adequate supply of each sort was an industrial enterprise with considerable material expenses.

The second difficulty is that unlike conventional text, mathematics is not linear in its construction: an equation depends on a two-dimensional arrangement of sorts. In the days of metal type, this was a considerable mechanical challenge, which demanded all the ingenuity the typesetter could muster. To create a really good piece of mathematical typography

"… gentlemen should be very exact in their copy, and compositors as careful in following it, that no alterations may ensue after it is composed; since changing and altering work of this nature is more troublesome to a compositor than can be imagined by one that has not tolerable knowedge of printing."

Caleb Stower, *The printer's grammar,* 1808

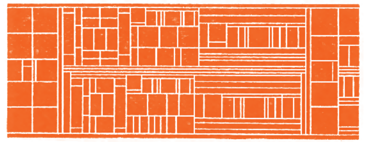
"Algebraic work, in treatises on algebra only, double price. When in lines or paragraphs, to be charged by time at the rate of 60 cents per hour, or about treble price."

Theodore L. De Vinne, *The printers' price,* 1871

the compositor must pay close attention to the spacing of the sorts in both dimensions. This would have been particularly difficult when adjacent sorts came from separate typefoundries and may be cast on subtly different sizes of body.

It is here that the far subtler matter of type design also enters the picture. A sort with a particular quirky design may be tolerated and readily understood as part of an otherwise ordinary word in most natural language typography: it may even add "color", enlivening the page of text. In mathematics it is necessary for the reader to be able to identify each and every sort reliably in isolation, and also to distinguish whether it is in italic, sans serif, bold or plain roman. A well-designed family of typefaces is a boon to mathematical typesetting and to the comprehension of the reader; "making do" with what is already available in types designed for typical text work is unlikely to be completely satisfactory, although that is exactly what happened for most of the history of mathematical printing.

$$\bar{x} = \frac{\int_0^a x\,dx \int_0^{\sqrt{a^2-x^2}} r\sqrt{x^2+r^2}\,dr}{\int_0^a dx \int_0^{\sqrt{a^2-x^2}} r\sqrt{x^2+r^2}\,dr} = \frac{2a}{5}$$



Handset mathematical type. Above, the equation as printed and, below, a diagram of the pieces of metal type and spacing material necessary. (From L.A.Legros and J.C.Grant: Typographical printing surfaces. Longman, Green, and Co., London, 1916.)

Hand typesetting of mathematics

The "invention of printing" in Europe around the year 1450 was really the invention of casting movable (reusable) type in metal: individual letters and signs were cast in hand-held moulds as raised surfaces on a conveniently sized rectangular stick of metal. These individual pieces of type were then assembled by hand into lines, and the lines into paragraphs and pages which were then printed. Once printed, the type could be reused to typeset fresh lines and pages. The earliest mathematical printing would have been from such type set by hand.

As mathematicians began to develop their subject, so they needed to find new notation to express their discovered knowledge. There is evidence of interplay between printer and mathematician, with the latter exploring what the former had to offer in the way of new type styles and symbols. In 1647, the English mathematician William Oughtred recognized the benefits of good notation in his *Algebra*:

> *Which Treatise being written not in the usuall synthetical manner, nor with verbous expression, but in the inventive way of Analitice, and with symboles or notes instead of words, seemed unto many very hard; though indeed it was but their owne diffidence, being scared by the newness of the delivery; and not any difficulty in the thing it selfe. For this specious and symbolicall manner, neither racketh the memory, nor chargeth the phantasie with comparing and laying things together; but plainly presenteth to the eye the whole course and processe of every operation and argumentation.*

The 'symboles or notes instead of words' were a considerable advance, a move from words that clearly referred to concrete things to symbols that represent abstractions. Oughtred used capital roman letters to denote his quantities, a practice that can be traced back to at least 1544 and the German mathematician Michael Stiffel.

× *into*, or *with*. The sign of multiplication, shewing that the quantities on each side the same are to be multiplied by one another, as $a \times b$ is to be read, a multiplied into b; 4×8, the product of 4 multiplied into 8. Wolfius and others make the sign of multiplication a dot between the two factors; thus, 7 . 4 signifies the product of 7 and 4. In algebra the sign is commonly omitted, and the two quantities put together; thus, $b\ d$ expresses the product of b and d. When one or both of the factors are compounded of several letters, they are distinguished by a line drawn over them; thus, the factum of $a + b - c$ into d, is wrote $d \times \overline{a + b - c}$. Leibnitz, Wolfius, and others, distinguish the compound factors, by including them in a parenthesis thus $(a + b - c)\ d$.

÷ *by*. The sign of division; thus, $a \div b$ denotes the quantity a to be divided by b. Wolfius makes the sign of division two dots; thus, 12 : 4 denotes the quotient of 12 divided by 4 = 3. If either the divisor or dividend, or both, be composed of several letters; for example, $a + b \div c$, instead of writing the quotient like a fraction.

involution. The character of involution.

evolution. The character of evolution, or the extracting of roots.

or are signs of majority; thus, a b, expresses that a is greater than b.

∠ or are signs of minority; when we would denote that a is less than b.

"Mathematical, algebraical, and geometrical characters" in J. Johnson, Typographia or the printers' instructor. *Longman et al., London, 1824.*

René Descartes, in *La Géométrie* of 1637, started the current practice of using italic letters to denote quantities of interest: printers had started to use italic type in the late fifteenth century, but not in conjunction with roman type. It was some time before many printers had roman and italic type cast to the same body size that would fit together conveniently.

As notation and calculations became more complicated, so the printer faced greater demands to assemble complicated expressions from individual pieces of type. They began to devise or suggest labor-saving notation:

> *… we propose one that is similar to an Italic l inverted, and whose figure takes in the whole depth of its body; which then would have the resemblance; viz.* 3/5 12/63 16/30.
>
> John Smith, *The printer's grammar,* 1755

In 1893, Oliver Heaviside popularized the work of James Clerk Maxwell in his *Electromagnetic theory,* and was the "discoverer" of another type style when seeking a notation for vectors:

> *Maxwell employed German or Gothic type. This was unfortunate choice, being by itself sufficient to prejudice readers against vectorial analysis. … Some of them are so much alike that a close scrutiny of them with a glass is need to distinguish them unless one is lynx-eyed. … Rejecting Germans and Greeks, I formerly used ordinary Roman letters to mean the same as Maxwell's corresponding Germans. … Finally I found salvation in Clarendons, and introduced the use of the kind of type so called I believe,*

∽ is the character of similitude used by Wolfius, Leibnitz, and others: It is used in other authors for the difference between two quantities, while it is unknown which is the greater of the two.

: : *so is*. The mark of geometrical proportion disjunct, and is usually placed between two pair of equal ratios, as 3 : 6 : : 4 : 8, shews that 3 is to 6 as 4 is to 8.

: or .·. is an arithmetical equal-proportion; as, 7 . 3 : 13 . 9; i. e. 7 is more than 3, as 13 is more than 9.

□ Quadrat, or regular quadrangle; as follows, □AB = □BC; i. e. the quadrangle upon the line AB is equal to the quadrangle upon the line BC.

△ Triangle; as, △ABC = △ADC.

∠ an Angle; as, ∠ABC = ∠ADC.

⊥ Perpendicular; as, AB ⊥ BC.

▭ Rectangled Parallelogram; or the product of two lines.

‖ The character of parallelism.

equiangular, or similar.

equilateral.

∟ right angle.

° denotes a degree; thus 45° implies 45 degrees.

′ a minute; thus, 50′, is 50 minutes: ″, ‴, ⁗, denote seconds, thirds, and fourths: and the same characters are used where the progressions are by tens, as it is here by sixties.

∺ the mark of geometrical proportion continued, implies the ratio to be still carried on without interruption, as 2, 4, 8, 16, 32, 64 ∺ are in the same uninterrupted proportion.

Johnson, ibid.

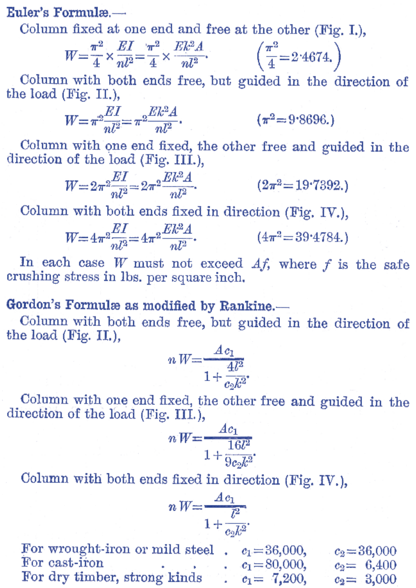
for vectors (Phil. Mag., August, 1886), and have found it thoroughly suitable.

Clarendon types were a relatively recent type style, characterized by low contrast between thick and thin strokes, heavy slab serifs and generous width. They provided a convenient solution to a notational problem, a solution now part of mathematical orthodoxy.

Sometimes a mathematician would come to the rescue of the printer. A bar over several characters is a very effective and easy-to-understand notation when symbols are to be linked, For purely mechanical reasons, though, it is difficult to place bars over or under characters in typesetting. Leibniz suggested a printer-friendly alternative:

> *When one or both of the factors are compounded of several letters, they are distinguished by a line over them; … Leibnitz, Wolfius, and others, distinguish the compound factors, by including them in parenthesis thus* ++$(a + b - c)d$.
>
> J. Johnson, *Typographia or the Printers' Instructor*, 1824

However, the effect of the mathematician raiding the printer's reserves of type, while providing solutions to notational problems, tended not to produce very harmonious typography. It can be difficult enough to express a mathematical idea in symbols without the visual distraction of those symbols being a rag-bag of ill-matching sorts.



Euler's Formulæ.—

Column fixed at one end and free at the other (Fig. I.),

$$W=\frac{\pi^2}{4}\times\frac{EI}{nl^2}=\frac{\pi^2}{4}\times\frac{Ek^2A}{nl^2}. \qquad \left(\frac{\pi^2}{4}=2{\cdot}4674.\right)$$

Column with both ends free, but guided in the direction of the load (Fig. II.),

$$W=\pi^2\frac{EI}{nl^2}=\pi^2\frac{Ek^2A}{nl^2}. \qquad (\pi^2=9{\cdot}8696.)$$

Column with one end fixed, the other free and guided in the direction of the load (Fig. III.),

$$W=2\pi^2\frac{EI}{nl^2}=2\pi^2\frac{Ek^2A}{nl^2}. \qquad (2\pi^2=19{\cdot}7392.)$$

Column with both ends fixed in direction (Fig. IV.),

$$W=4\pi^2\frac{EI}{nl^2}=4\pi^2\frac{Ek^2A}{nl^2}. \qquad (4\pi^2=39{\cdot}4784.)$$

In each case W must not exceed Af, where f is the safe crushing stress in lbs. per square inch.

Gordon's Formulæ as modified by Rankine.—

Column with both ends free, but guided in the direction of the load (Fig. II.),

$$nW=\frac{Ac_1}{1+\frac{4l^2}{c_2k^2}}.$$

Column with one end fixed, the other free and guided in the direction of the load (Fig. III.),

$$nW=\frac{Ac_1}{1+\frac{16l^2}{9c_2k^2}}.$$

Column with both ends fixed in direction (Fig. IV.),

$$nW=\frac{Ac_1}{1+\frac{l^2}{c_2k^2}}.$$

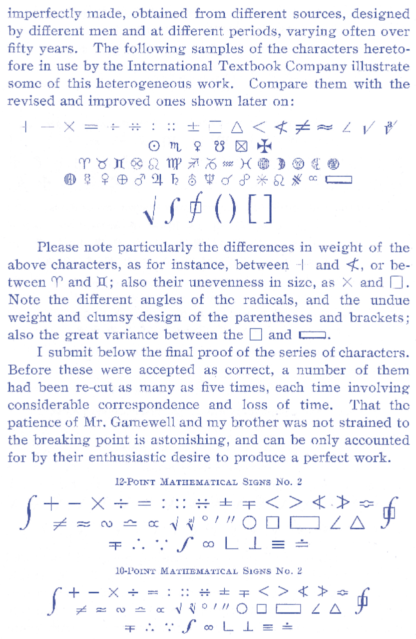
For wrought-iron or mild steel .	$c_1=36{,}000$,	$c_2=36{,}000$
For cast-iron . . .	$c_1=80{,}000$,	$c_2=6{,}400$
For dry timber, strong kinds .	$c_1=7{,}200$,	$c_2=3{,}000$

Typical handset maths from D. A. Low, A pocket-book for mechanical engineers. *Longmans, Green and Co., London, 1915.*

Improving mathematical typography

One early attempt to improve mathematical typography involved the Inland Typefoundry of St Louis in 1900. At the suggestion of a publisher they cut punches and made sorts to match their existing Oldstyle No.11 type. They also rationalized the setting of sorts to uniform widths and the alignment of different typefaces on the body of the type. These innovations made the mechanical work of the hand compositor much easier. It is therefore unclear why more printers and publishers did not take advantage of this work, but it was largely neglected.

A more successful attempt involved Oxford University Press and the Monotype system for composing metal type mechanically. Following an audacious raid by the Oxford University Press into the "enemy territory" of Cambridge in March 1928, OUP's printer, John Johnson, was presented with the problem of setting mathematics for a newly founded physics series and its first Cambridge-based author, P. A. M. Dirac. A pair of mathematical journals acquired at about the same time from a small Cambridge publisher and previously set in oldstyle types provided a training ground. Johnson rose to the challenge and set about researching a typeface to use. His choice was Monotype's Modern Series 7, possibly because OUP already had a large investment in modern types and matching Greeks and special sorts for its dictionary work. Johnson did not just



imperfectly made, obtained from different sources, designed by different men and at different periods, varying often over fifty years. The following samples of the characters heretofore in use by the International Textbook Company illustrate some of this heterogeneous work. Compare them with the revised and improved ones shown later on:

Please note particularly the differences in weight of the above characters, as for instance, between ⊣ and ≮, or between ♈ and ♊; also their unevenness in size, as × and □. Note the different angles of the radicals, and the undue weight and clumsy design of the parentheses and brackets; also the great variance between the □ and ▭.

I submit below the final proof of the series of characters. Before these were accepted as correct, a number of them had been re-cut as many as five times, each time involving considerable correspondence and loss of time. That the patience of Mr. Gamewell and my brother was not strained to the breaking point is astonishing, and can be only accounted for by their enthusiastic desire to produce a perfect work.

12-Point Mathematical Signs No. 2

10-Point Mathematical Signs No. 2

Inland Typefoundry's improvements to oldstyle type for mathematical typesetting (excerpt from Old style type on bookwork, *privately printed, St Louis, 1902.)*

commission new mathematical sorts to match the existing holdings of OUP, he also sought to formalize the rules of mathematical typesetting. With the assistance of a small panel of mathematicians, he codified the rules of mathematical typesetting. These were eventually published by Johnson's successor, Charles Batey in *The Printing of Mathematics* [T.W. Chaundy *et al.* 1950]. More than half of the Oxford rules are given over to matters of spacing, emphasizing the great importance in mathematical typesetting not just of choosing the correct sort, but also its correct disposition in relation to others.

> "Printing-house proprietors and authors, from force of circumstances, have had to be content with many characters imperfectly made, obtained from different sources, designed by different men and at different periods, varying often over fifty years."
>
> Carl Schraubstadter, *Old style type on bookwork,* 1902

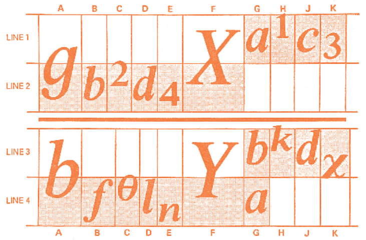
The Oxford rules were developed for use with Monotype keyboards and casters. This was the first mechanization of mathematical typesetting. Even though many elements of an equation could be keyboarded in this system, the assembly of the elements required skilled handwork. But with good training and the inherent precision and controllability of the Monotype machinery, the results were exceptionally good compared to what went before. A new standard of mathematical typography had been set.

$$
\begin{aligned}
\overset{v}{d}\overset{0}{R} &= \overset{v}{d}\int_{\tau_0}^{\tau} \mathfrak{L}\, d\tau = \int_{\tau_0}^{\tau}\left(\frac{\partial\mathfrak{L}}{\partial q^{\phi}}\,\overset{v}{d}q^{\phi} + \frac{\partial\mathfrak{L}}{\partial \overset{*}{q}{}^{\phi}}\,\overset{v*}{d}q^{\phi}\right) d\tau \\
&= \int_{\tau_0}^{\tau}\left\{\left(\frac{d}{d\tau}\frac{\partial\mathfrak{L}}{\partial \overset{*}{q}{}^{\phi}}\right)\overset{v}{d}q^{\phi} + p_{\phi}\,\overset{v*}{d}q^{\phi}\right\} d\tau \\
&= \int_{\tau_0}^{\tau}\left(\overset{*}{p}_{\phi}\,\overset{v}{d}q^{\phi} + p_{\phi}\,\overset{v*}{d}q^{\phi}\right) d\tau = \int_{\tau_0}^{\tau} d\left(p_{\phi}\,\overset{v}{d}q^{\phi}\right) \\
&= (p_{\phi})_{\tau=\tau}\,\overset{v}{d}q^{\phi} - (p_{\phi})_{\tau=\tau_0}\,\underset{0}{\overset{v}{d}q^{\phi}} \quad (\phi = 0, 1, \ldots, n),
\end{aligned}
$$

Monotype Modern Series 7 type employed by OUP for mathematical typesetting.

The skilled handwork of the Oxford method made it slow and expensive both in operation and in training. With an increase in scientific publishing and in mathematical work in particular, the Monotype Corporation started to investigate ways to reduce handwork and increase what could be accomplished directly from the keyboard. Using the American Patton Method as inspiration, the Monotype 4-line system was born. The highly popular Times typeface was adapted

as Series 569. The parent Times face was well provided with a range of sorts, but the peculiar spacing requirements of mathematics and the highly innovative technicalities of the 4-line system made special demands. At the system's heart, a 10 point face was cast on a 6 point body and each equation was broken down into four lines. With the introduction of the Monotype 4-line system in 1959 hot-metal typesetting had reached its "technological zenith" [David Saunders, *The Monotype Recorder,* New Series 10, 1997]. This and the previous Oxford system provide benchmarks that are still used today.



The details of Monotype's 4-line system for setting mathematics.

The eclipsing of hot-metal systems

Mathematicians and scientists may have been well-served typographically by Monotype, but increasingly they wanted a less expensive way of composing mathematics that was more directly under their control. Publishers also wanted an inexpensive alternative method to compose mathematics in order to make more specialist and ephemeral publications viable. Technology new to printing was pressed into service. The IBM golfball typewriter and Varityper both offered strike-on typesetting with interchangeable type heads. The interchangeability gave access to the range of special sorts needed; good typists or the mathematicians themselves could operate the machine, reducing labor charges dramatically. Huge compromises had to be made in typographical quality, with very limited variations in character widths

To evaluate $I_2^{(m)} = \int_{x_h}^{1} J_2^{(m)}\, dx$, it is noted that

$$
\lim_{x\to 1} J_2^{(m)} = \frac{-A\sqrt{a_0}}{4(1-x_h)(1+\lambda_i^2)} \cdot \frac{(1 - x^* \cos\sigma_m)}{(1 + x^{*2} - 2x^*\cos\sigma_m)^{3/2}}\left(\frac{a_1}{2a_0} - \frac{b_2}{b_1}\right)
$$

Typewriter composition of mathematics.

and spacing, but immediacy and cost reduction were important driving forces.

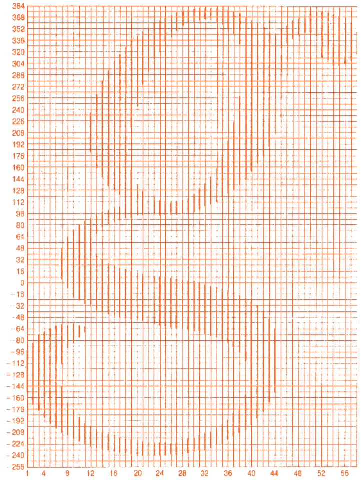
Phototypesetting, which had first been developed in the 1940s, started to appear as an common method of composition in the 1960s as offset lithographic printing became more popular. Photosetting machines generated lines of type from film masters, using a photographic process to set the type on rolls of paper or film from which lithographic plates could be made. Most early photosetters were limited in the number of sorts carried on their film masters, making them unsuitable for anything but the simplest mathematical work. An exception was the Monophoto machine from Monotype. Its film masters were carried in individual and interchangeable photomatrices. Coupled with an adaptation of the 4-line system, great flexibility and good quality were achieved for mathematical setting.

The advent of photocomposers with digital rather than photographic masters opened a new era in mathematical typesetting. Digital storage meant that a huge range of special sorts could be accessed easily and created almost as easily. Digital storage also meant computer

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}k\, b(k,t)\,\mathrm{e}^{-\mathrm{i}k(x+y)}\left(\frac{1}{2\pi}\lim_{\epsilon\to 0}\int_{-\infty}^{\infty}\mathrm{d}k'\frac{H(k',x)\,\mathrm{e}^{-\mathrm{i}k'x}}{\epsilon-\mathrm{i}(k'+k)}+\sum_{n=1}^{n}\frac{G_n(x)}{\kappa_n-\mathrm{i}k}\mathrm{e}^{\kappa_n x}\right)$$
$$+\sum_{n=1}^{N}c_n^2(t)\,\mathrm{e}^{\kappa_n(x+y)}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}k\frac{H(k,x)}{\kappa_n-\mathrm{i}k}\mathrm{e}^{-\mathrm{i}kx}+\sum_{m=1}^{n}\frac{G_m(x)}{\kappa_m+\kappa_n}\mathrm{e}^{\kappa_m x}\right).$$

Linotron composition of mathematics.

control. That computing power could be harnessed to some of the trickier problems of mathematical spacing and layout in specialist programs and routines. The Linotron 404 and, more importantly, 202 machines from Linotype used a high-definition cathode ray tube to project a digitally generated image onto photosensitive paper or film under microprocessor control. The 202 set the standard for quality mathematical composition from its introduction in 1978.



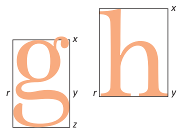
Digital structure of type used in the Linotron 202.

TEX

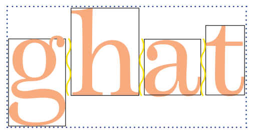
Despite these advances in professional typesetting, the poor typographical standards of the populist strike-on composition so disgusted one mathematician that he seized the moment, and the new technology of raster-based laser output, to completely change how mathematicians work. Donald Knuth developed TEX as a word-processing and typesetting system capable of mathematical work. Intriguingly, Knuth's explanation of the working principles in terms of boxes and glue, the boxes being the sorts and the glue variable amounts of space, shows similarity to the physical workings of the Monotype machine, with which Knuth was familiar. The fixed-size of a box dictates a certain size for the character to occupy the box (or vice versa). For TEX to operate, it needs information about the chosen typeface. Knuth was thorough in his research and also developed a program for drawing typefaces, Metafont. The typeface he chose to run with TEX was

Computer Modern, similar to the Modern Series 7 of traditionally set mathematics books. TeX was a resounding success with mathematicians and scientists for a variety of reasons, not least of which was being free software. TeX quickly developed a base of very active and devoted users who cooperated to improve and extend it, and it produces results of remarkable quality and consistency which can be reliably reproduced on a variety of printers. It also allows data entry in a “mathematically logical” order when building equations: an advantage over traditional typesetting programs, which tended to require data entry to follow the mechanical constraints of the technology and which made no mathematical sense. Printers were initially very slow to make use of its potential: it bypassed all their existing systems and its data-entry order was alien to them. But TeX’s overwhelming success with authors means that mathematical typesetters have had to learn how to use it and apply their accumulated typographic and design experience to it.

Starting, as Knuth did, with no more than access to digital raster image setters and computing power, he was free to work from scratch without constraints of any pre-existing programs or systems. This allowed him to build a logical and very robust structure aimed at solving the problem, and TeX is certain to influence all future developments in mathematical typesetting.



Typographic elements in TeX; are defined as boxes. the x-y distance is the glyph height, the y-z is its depth. Where the two meet is the reference point (r).



TeX character boxes are ‘glued’ together to form word boxes (dotted line).

TeX word boxes are glued together to form line boxes (double-line) and line boxes are glued together to form paragraph boxes. The ‘glue’ between any of these boxes can be stretched or compressed to alter character spacing, word spacing etc. and to aid in paragraph justification.

The Cambria Math font

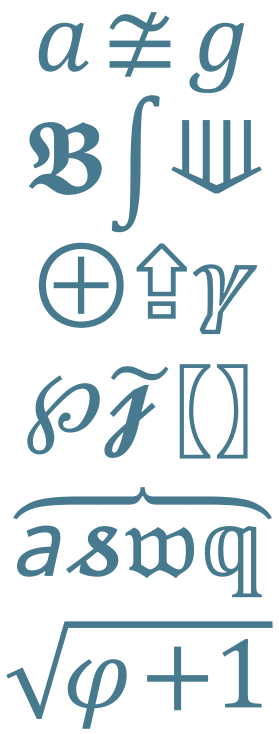
As the previous section records, the history of mathematical typesetting has often mingled innovation with expedience, finding new uses for existing typographic elements and extending existing typefaces to incorporate specialised symbols. In this aspect, Microsoft has broken with tradition. While the Inland Typefoundry, Oxford University Press and Monotype historically sought to improve the typographic quality of mathematical setting by adapting existing types, Microsoft has commissioned a new typeface, Cambria Math, designed from the outset with the needs of mathematics in mind. The new type was also designed to perform well on screen and to leverage the benefits of Microsoft’s acclaimed ClearType® rendering, acknowledging the increasing importance of electronic publishing and information exchange.

The following section looks at key features of the Cambria Math font, including extensions of the OpenType Layout format and new font tables to support mathematical typography.

Features of the Cambria Math font

Custom type for mathematical typesetting is nothing new. Since its early days, the setting of mathematics has posed problems for the typesetter that have been answered by the manufacture of new typographic characters. The inventiveness of mathematicians and scientists in devising new symbols to express novel concepts has placed considerable demand on the makers of type, although the cost of manufacturing weighed against the relatively small returns on specialist publishing has limited the actual number of fonts available for mathematical work. The advent of computer typesetting for mathematics has not, so far, changed this situation, and if anything the variety of typefaces available is reduced, since most mathematical texts have been produced in application- and font-specific formats.

The recent inclusion of a large number of characters for mathematical publishing in Unicode and corresponding ISO standards, the specification of the MathML markup language by the World Wide Web Consortium, and the inclusion of mathematical typesetting tools in mainstream software such as Microsoft Office, opens the door to broader access and increased exchangeability of mathematical texts. These welcome developments also invite the design of new typefaces to facilitate the typesetting of mathematics in new environments and for new media.



The Cambria Math font is an extension of Cambria, a four-style font family designed by Jelle Bosma as part of the Microsoft ClearType Font Collection. Cambria, like other members of the collection, was designed to take advantage of Microsoft's ClearType rendering system, which is designed to improve the experience of screen reading. The weight and proportion of the letters, numerals and symbols in Cambria have been carefully designed to be clearly legible at the small sizes and low resolutions of the screen environment, enhanced by ClearType's subpixel rendering and positioning.

Development of the Cambria Math font involved design of additional glyphs (the visual representations of the abstract characters encoded in text), revision of some of the existing glyphs from the original Cambria, and the inclusion of advanced information in the font to be used by the Microsoft math layout handler. This information includes new, math-specific features within the OpenType Layout architecture already used for supporting complex scripts (*e.g.* Arabic) and high quality typography, and addition of a newly specified MATH table in the font containing values to be interpreted by the math handler during layout.

The values in the MATH table govern positioning, preferred scaling factors and substitution of glyph variants, *e.g.* for growing parentheses or braces; they are font-specific and are set by the type designer or font technician. Additional, non font-specific rules for positioning and spacing are incorporated into the math handler, which also has the capability to make positioning decisions by analysing the glyphs presented to it.

For more information about the ClearType Font Collection and the ClearType rendering system, see the Microsoft publication *Now read this.*

For general information about OpenType, see the Microsoft Typography website: *www.microsoft.com/typography/*

Character support and glyph set

The character repertoire for Cambria Math was established by the mathematical characters which have been defined in the Unicode standard. These include a large number of operators and other symbols as well as alphanumeric characters in the wide range of style encountered in mathmatical text (bold, italic, script, fraktur, sans serif, etc.). In Cambria Math, an extensive subset of these characters were added to the pan-European multilingual character set of the original Cambria fonts, enabling the seamless intergration of mathematical settings into texts in a wide range of languages written in the Latin, Greek and Cyrillic scripts.

In addition to the characters defined in Unicode, there is also a large number of unencoded variant glyphs, which are accessed through OpenType Layout features or MATH table rules. Most of these are variant sizes of a base character for stretching around large equations or over and under strings of text. For instance, there are several sizes of brackets, parentheses and radicals, as well as accents of different widths that can sit over wide characters or over multiple characters. Other unencoded glyphs include scaling forms tuned for use as superscripts or subscripts, and dotless versions of *i* and *j* characters for use with accents above.

The original Cambria Regular font contained 992 glyphs, already making it a "large font" by many standards. The Cambria Math font contains an additional 2,900 glyphs.

For more information about the mathematical characters in Unicode, refer to Unicode Technical Report 25: *www.unicode.org/reports/tr25/*

Overview of the MATH table

TrueType and OpenType are examples of "sfnt format" fonts, which are made up of a number of discrete tables, some required and some optional. This format is easily extensible by the addition of new tables.

The new MATH table communicates between the math font and the math handler. It contains data related to horizontal spacing, vertical positioning, glyph variants and assemblies, preferences for elements such as rules which are "drawn" by the handler, special math kerning, and additional global parameters. These are too many to be exhaustively detailed in this small booklet, but many of them are illustrated in the pages that follow and give some idea of both the general methodology and the level of control available to math font developers.

The information in the MATH table can be categorized into font-level values, glyph-level values, and glyph lists. Font-level values are stored as "constants", in which the font developer is able to set more than fifty parameters related to *e.g.* superscript and subscript positioning; linespacing within stacked equations; axis height for vertical alignment; bar, rule and vinculum thickness; gap allowances between elements such as text and bars; and many others.

Glyph-level values apply to individual glyphs and include italics correction, which helps refine positioning of superscripts; accent attachment, which allows the developer to define preferred positioning of accents (rather than relying on automatic centering); and math

A selection of math constants:

Math Leading
Axis Height
Accent Base Height
Flattened Accent Base Height
Subscript Shift Down
Subscript Top Max
Subscript Baseline Drop Min
Superscript Shift Up
Superscript Shift Up Cramped
Superscript Bottom Min
Superscript Baseline Drop Max
Sub-Superscript Min Gap
Superscript Bottom Max With Subscript
Space After Script
Upper Limit Gap Min
Upper Limit Baseline Rise Min
Lower Limit Gap Min
Lower Limit Baseline Drop Min
Stack Top Shift Up
Stack Top Display Style Shift Up
Stack Bottom Shift Down
Stack Bottom Display Style Shift Down
Stack Gap Min
Stack Display Style Gap Min
Stretch Stack Top Shift Up

kerning, which gives fine control over relative horizontal spacing of glyphs at different vertical alignments.

Glyph lists provide character variant and assembly information. In simple terms, these indicate a base Unicode character and a list of variants for that character and/or ways to assemble the character from multiple glyphs. The variant list is accessed by the math handler when determining the best glyph to use in a given situation. For instance, as a fraction or stack grows in the vertical direction and is enclosed in a delimeters, *e.g.* parentheses, taller versions of the delimeter can be used. The math handler measures the height or width of an equation and then selects the best match the variant list or assemblies list. [See pages 29–30 for more information about variants and assemblies.]

Size-specific OpenType script-styles

When a letter or numeral is mechanically scaled, for example for use as a superscript, the strokes that make up the glyph are reduced in weight, while the internal spaces of the form are pinched. It is preferable for glyphs to become more robust as they are reduced in size, and for their proportions and spacing to become more generous, so that they harmonise well with nearby full-size glyphs.

In Microsoft math fonts, size-specific “script” and “script-script” variant glyphs are designed at full size relative to the base Unicode characters they represent, and are then scaled for use as super or subscript forms according to factors specified in the MATH table font constants. When displaying a character as script or script-script, the math handler calls a new OpenType Layout feature, <ssty>, which performs a glyph substitution of the appropriate size-specific glyph. The <ssty> feature employs an enumerated lookup type, so the same feature may include both script and script-script variants, mapped as the first and second alternates respectively.

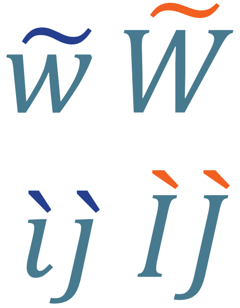
The adjustments to weight and proportion desirable for the script and script-script styles are not necessarily linear relative to the base glyph. In the case of Cambria Math, the ascenders and capital height of the script variants were shortened so exponents would be more compact and work better for inline formulae, but in the script-script style they are taller to aid legibility at very small sizes.

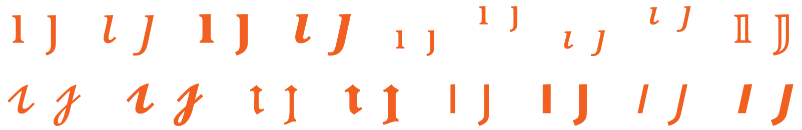
Flattened accents and dotless forms

Two additional new OpenType Layout features have been introduced for mathematical typesetting: one is Flattened Accents <flac>, the other is Dotless Forms <dtls>.

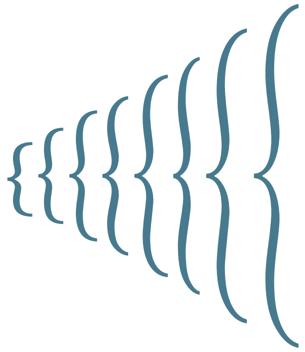
The <flac> feature accesses variant accent forms which have been designed to sit above capitals or other tall letters. These accents are vertically shorter than regular accents designed for use with x-height lowercase letters, so allow for more compact linespacing and vertical gaps in stacked equations. The height at which flattened accents are deployed by the math handler is set in the MATH table and the <flac> feature lookup simply maps accents which have a "flattened" variants, as shown at right.

The <dtls> feature maps dotted characters *i* and *j*, in all their forms, to variants with no dot. These are used when an accent needs to be placed above the character. Arbitrary accent positioning over base glyphs is controlled via the 'accent attachment' values set in the MATH table glyph-level values; if no accent attachment values are set, then the accent is centered over the base glyph.





Variants and assemblies

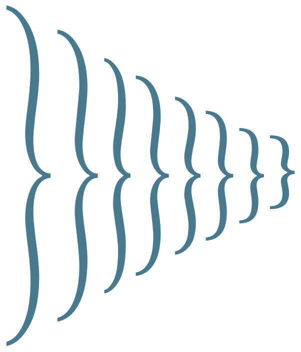


The math font format allows the designer to define sets of both vertical and horizontal glyph variants. These are designed and then listed in the MATH table as variants of a base glyph.

Vertical variants are most often used to "grow" delimiters (parentheses, braces, brackets, etc.) or radical signs, which should match the height of expression they contain. Horizontal variants can be defined for characters such as over- and under-delimiters and arrows or vectors, *e.g.* $\underbrace{abcd}$. They can also be used to extend an accent over glyphs of various widths or over multiple glyphs, *e.g.* $\overset{\frown}{ax}$.

The math handler measures the width or height of a given expression, then checks the MATH table glyph lists and selects the variant whose height or width most closely corresponds to the size of the expression. The Cambria Math font contains up to eight different variants for some characters.

If the math handler determines that all the available variants are too small, it then proceeds to assembly information for the character. Assemblies are sub-character glyphs, *i.e.* two or more "pieces" that are assembled by the



math handler to represent the character that needs to be resized to fit the expression.

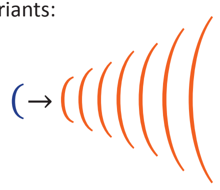
Assemblies typically consist of top and bottom or left and right pieces, and medial sections. They are designed to align with one another and to smoothly connect. The connections and medial sections are necessarily straight, so elements built from assemblies, especially braces, lack the elegant curvature of the variant forms, but they are very versatile and because sections may overlap they can adapt very closely to the size of an expression.

The assembly pieces are listed in the MATH table lists for a given character and additional information is provided to specify how much overlap one piece can have with its adjoining piece. The math handler references this information and adjusts the positioning of the pieces to match the height or width of an expression.

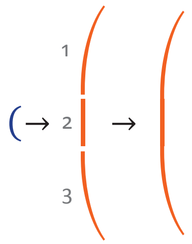
Variants:
A Unicode character, such as a parenthesis:

(

is mapped to a series of glyph variants:



Assemblies:
If the largest of the variants is not tall or wide enough, an assembly can be constructed from a series of individual parts:



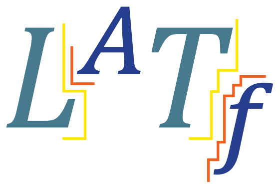
Math kerning

In typical text setting, adjustments to improve spacing by moving glyphs closer or further apart, called kerning, generally involve only horizontal relationships. These adjustments can be easily managed by a single kerning value applied whenever two particular glyphs are next to one another. In mathematical setting, the relationship between adjacent glyphs may involve complex horizontal and vertical relationships, especially when glyphs are scaled and positioned for use in superscript or subscript roles.

Microsoft's math font format uses "cut-ins" to enable superscript and subscript glyphs to nestle against adjacent glyphs while ensuring that sufficient distance is maintained. The cut-ins are stepped, so that as glyphs are moved vertically relative to each other an appropriate horizontal relationship is maintained.

Cut-in values are defined by the font developer in the MATH table and may be specified for each quadrant: upper-left, upper-right, lower-left and lower-right. This allows for optimal spacing in relationships between any adjacent quadrant on two glyphs, *e.g.* upper-right to lower-left, as in a base to superscript relationship. The cut-in values can be either positive or negative, so may move glyphs closer together or further apart.

Where two sets of cut-ins meet, they may interract, affecting the spacing. In the illustrations here and overleaf, the yellow lines indicate



right-side cut-ins on the base glyph, and the orange lines indicate left-side cut-ins on the corresponding quadrants of script style glyphs.

Each quadrant may have multiple cut-ins, defined at different heights, and the cut-ins may be as coarsely or finely defined as the font developer deems to be appropriate, taking into account the font constants that determine scaling and vertical positioning rules for glyphs. Some glyphs may have only a single cut-in, while others may have many, creating an envelope around the glyph shape or, if desirable, allowing part of the outline to extend beyond the cut in (as in the illustration on the preceding page, where the descending tail of the subscript *f* pierces the edge of the cut-in).

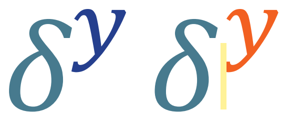
Since the vertical position of superscript and subscript glyphs, initially defined in the MATH table font constants, may be dynamically adjusted dependent of the contents of a formula or equation, cut-ins provide flexible control of horizontal relationships *based on glyph shapes.* In the example of the superscript *A* in the illustrations, if it were desirable to lower the superscript it would encounter a change in the cut-in envelope, which would prevent it from colliding with the terminal of the *L* (as shown at right).

Each cut-in can also have device correction values; *i.e.* size-specific adjustments to ensure optimal relationships between glyphs even at low resolutions on screen, where rounding errors on the coarse pixel grid might otherwise cause glyphs to be too close or too far apart.

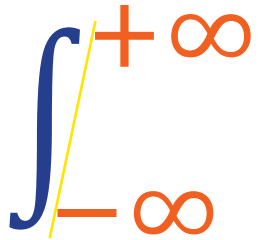
Italics correction

The angle of italics or other slanted glyphs may result in them appearing too close to or even colliding with following upright or superscript glyphs. In order to ensure that a reasonable minimum distance is maintained, the MATH table contains glyph-level values for “italics correction”. The italics correction value is added to the advance width of the glyph when it is followed by an upright character, *e.g.* a delimeter or operator, or by a superscript.

Italics correction is also used for positioning limits on *n*-ary operators, particularly integrals, as shown at right. Note that in this case, horizontal adjustment is defined according to the italics corrections value rather than by the cut-in kerning used for superscript and subscripts relative to base glyphs. Because the integral sign can grow and limits may be more complex than shown here, involving more than one line and requiring dynamic vertical adjustment, the italics correction value provides a flexible mechanism to ensure a consistent relationship between the limits. The position of the upper limit is determined by shifting the limit to the right by one half of the italics correction value of the integral glyph, and the lower limit is shifted to the left by the same amount. In this way, a consistent angle is maintained.



Left, without italics correction; right, with italics correction.

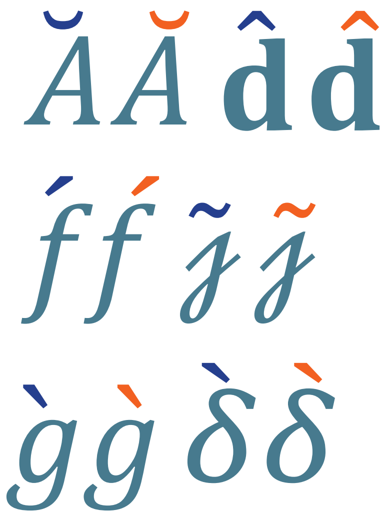


Accent attachment

Base characters in mathematical typesetting may carry accents, which may not correspond to diacritics in natural language orthographies, and which need to be correctly positioned relative to the base. This is particularly important in the case of italics, when simply centering the accent glyph on the advance width of the base result in the accent being too far left, far from its ideal location relative to the top of the base glyph.

Accent attachment is specified in the math table on a glyph-to-glyph basis. If a pair of base and accent glyphs do not contain an accent attachment entry in the table, then the math handler defaults to centering the accent on the glyph. Since the absolute center of a glyph space and the optical center of a glyph seldom match exactly, and because so many glyphs have asymmetric shapes that call for the accent to be off-center, most glyphs will have accent attachment positions defined.

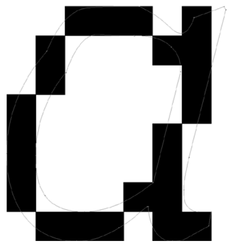
The illustration contrasts centered accents, blue, with accents positioned using accent attachment values, orange. Note that the accent attachment interacts with the flattened accents and dotless forms features discussed on page 28.



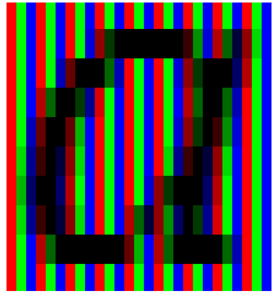
Cambria Math on screen

ClearType is a Microsoft text rendering technology for screen which takes advantage of discreet red, green and blue subpixels in liquid crystal displays (LCDs). To render text on screen, glyph outlines in the font must be “rasterised”, *i.e.* fitted to a grid of pixels to give the best representation of the glyph shape at a given size and resolution. At the small sizes normally used for text, and in the relatively low resolution of screen displays as contrasted with print media, the pixel grid is very coarse. With traditional black and white or greyscale rendering, it is often impossible to render details or even give a reasonable impression of a particular typeface design. ClearType significantly improves the display of text on screen by addressing individual subpixels during rasterisation, effectively tripling the resolution in one direction (usually horizontal). At the same time, ClearType uses sophisticated color filtering to maintain consistent stroke color, even for thick and thin strokes of variant weight, providing a cleaner and stronger glyph image than older greyscale font smoothing technologies.

The rapid growth and impact of the Internet in all areas of life in the past ten years has greatly increased the amount of text that we read on computer screens rather than in print. Mathematicians and scientists, of course, were among the earliest users of the Internet and electronic communications, but only fairly recent developments—the inclusion of specialist mathematical characters in Unicode and the specification



An italic letter hinted for black & white display. The outline is distorted by hint instructions to turn on or off specific pixels.



ClearType rendering using individual subpixels results in smoother curves and more natural diagonal strokes.

$$v = V\left[1 - \sum_{s=1}^{\infty} \frac{\pi J_1^2(b\beta_s)}{[J_1^2(b\beta_s) - J_0^2(a\beta_s)]}\{J_0(b\beta_s)Y_0(a\beta_s) - Y_0(b\beta_s)J_0(a\beta_s)\}\, exp(-\kappa\beta_s^2 t)\right]$$

of the MathML markup language—have made possible the reliable interchange of mathematical content in electronic documents. Cambria Math is the first font developed specifically to provide high quality display of mathematical typesetting in a screen environment. This goal has guided all aspects of the font production, from the design of glyph outlines to the specification of MATH table values.

By leveraging the display improvements and positioning refinements of ClearType, even scaled glyphs in math settings, *e.g.* superscript and subscript forms, can be cleanly rendered at typical text sizes. Glyph rendering is assisted by “hints” in the font: instruction sets for each glyph that ensure the rasterizer maintains consistency of stroke weights and proportions in the low resolution grid. In addition to these general glyph hints, Cambria Math contains device-dependent adjustments to values in the MATH table, *e.g.* desired kerning cut-in depths and heights for specific sizes.

Above: a screenshot of a 12pt equation on a 145 pixel per inch display. It is difficult to accurately represent screen rendering in print, but this illustration, using full pixel color representation of the ClearType subpixel rendering, gives some idea of the clarity and legibility of the Cambria Math font on screen.

Input and layout

An examination of the examples in the previous sections gives some idea of the difficulties involved in mathematical layout and the intricate solutions required. Having looked at these issues from the perspective of the font tables and features, we can now look at how math layout is performed in the larger context of Microsoft applications, and at how the user creates mathematical content.

Describing math layout in all its complexities is beyond the scope of this publication, as is full documentation of the new math input language. Most of the material in the following section is introductory, and provides a basic overview of Microsoft’s approach to mathematical input and layout. The section concludes with a more technical discussion of the layout mechanisms employed in correctly typesetting a famous equation.

Mathematical input and layout

Microsoft's math layout depends on Unicode encoding and follows the lead of TEX and MathML 2.0 in key areas. It is performed as a collaboration between four entities:

- a Unicode rich-text processing program such as Microsoft Word
- the math handler built into the latest version of the Microsoft text layout component
- the math font
- the math font handler

This collaboration is invoked whenever text inside a designated math zone needs to be displayed. A math zone is created in a document by the user, and is distinguished from regular text for layout purposes. All text in the math zone is rendered using appropriate glyphs (which may vary from related, non-math forms, as shown at right) and according to measurements dependent on glyph ascents, descents, and widths, as determined by the math handler rules and the contents of the font MATH table.

Within the math zone, text is entered using a special input language. This is intuitive and easy to learn, and allows the user to create mathematical content quickly while relying on the layout engine and font to ensure accurate glyph display, arrangement and spacing.

adfhiknvxy

Cambria Italic glyphs

$adfhiknvxy$

Cambria Math math-italic glyphs

The italic letters used for math variables in the Cambria Math font differ from the regular Cambria Italic font letters in both form and spacing. The math italics are encoded as Unicode math alphanumeric characters and are stored in the same font as the regular, upright characters and other alphanumeric forms such as blackletter, doublestruck, etc.

The math input language

In the simplest cases, such as an equation like $a = b + c$, the variables a, b and c are represented by Unicode math-italic letters and the operators are separated from the letters by spacing rules according to established math typesetting conventions (as documented by Donald Knuth in *The TEXbook*.) The input experience for the user in such a case is much simpler than the layout requirements might imply: he simply types the normal key sequence `a=b+c` in the math zone and the math handler takes care of the rest, converting the letters to the appropriate math-italic characters from the math font and applying the correct form and spacing for the operators.

Input of characters in a math zone is achieved via a linear or "nearly plain text" input language. This linear language is a means of entering specialized characters into text by typing an escape sequence, usually a backslash `\` followed by a keyword *e.g.* `\alpha` will display α; `\sum` will display $\sum$, and so forth. The ASCII characters on a standard keyboard are entered directly and autocorrected as required. *Note that these keyboard characters are used only for input notation: the resulting text is stored using the appropriate Unicode math characters.* Some keyed characters are used as switches or triggers; for instance, a forward slash `/` indicates that a fraction is to be constructed from what comes before and after the slash. An underscore character `_` indicates a subscript, and a circumflex `^` indicates a superscript. The space character also has a role, delimiting

In a math zone, the user types:

a=b+c

As each character is entered, the application substitutes the appropriate math italic character and makes appropriate adjustments to the spacing.

$$a = b + c$$

In a math zone, the user types:

a/(b+c) `[space]`

As each character is entered, the application substitutes the appropriate math italic character. The space character triggers the fraction object layout and positioning:

$$\frac{a}{b + c}$$

the operands of the linear format notation. The space character indicates that preceding elements are to be built-up, *e.g.* a_2 followed by the space character would render a_2. As well as keyboarding, other means of input are also possible, such as pull-down menus or handwriting recognition, on a Tablet PC for instance.

Full details on the math input language can be found in Unicode Technical Note 28: *www.unicode.org/reports/tn28/*

The math layout handler

The collaboration between the math layout handler and its text-processing client, *e.g.* Microsoft Word, is carried out via a library of layout methods and rules, along with extensive communication between the math handler and the other layout components. The math handler uses a "callback" mechanism to ask for information about the text, such as where a run of text begins and ends, the properties and content of that run, the glyphs and glyph dimensions to be used, the math-object properties, line breaking data, etc. This results in callback functions that, for instance, examine the characters in the document's backing store of characters or ask the math-font handler and operating system to obtain data to assist layout decisions.

For all this to work correctly, the client application needs to provide a backing store with the correct function names and arguments; the correct bases, subscripts or superscripts; the correct integrands, summands; etc. This means that the input model requires some understanding of the underlying mathematics. This is a key difference between computer typesetting of mathematics and a purely visual composition as practised in the days of metal type. It is not enough to look at a printed or manuscript equation to know how to reproduce it from the available glyphs in a font: it is necessary to understand at least something of what the equation means and how the parts of the equation relate to one another.

Although this requirement of the input model was introduced to enable aspects of typographical layout for mathematics, it is proving helpful in interacting with mathematical calculation engines, allowing for close integration between calculation and layout.

Math zone text is described in a mark-up language, an XML, called OMML (Office MathML). OMML can be converted to and from MathML 2.04 and contains features of both the MathML presentation and content tag sets. OMML can be embedded in other XMLs such as WordML, and vice versa.

Math objects and spacing

The layout of a linear equation such as $a = b + c$ is relatively easy. In more complicated equations, special built-up math handler objects are used to place the glyphs in the correct places. The math handler uses an extended set of spacing rules for operators and math objects to automate a number of spacing refinements that TEX delegates to the user. The built up math objects are summarized in the table opposite.

The function-apply object, for example, is used to insert proper spacing around sin and x in the expression sin x. The n-ary object is used to insert proper spacing around the n-ary symbol and around the "n-aryand", *e.g.* integrand or summand. The subscript and superscript objects are used to provide the proper kerning between the base and the script element, using the cut-in values in the font MATH table.

The objects are based on a detailed analysis of the requirements of laying out mathematical notation, and provide the math handler, working with the math font, with precise control over the positioning of every element of an equation. Individual characters and smaller objects are positioned relative to each other and built up into larger objects that, in turn, are positioned relative to adjacent objects as appropriate to the overall shape of the equation.

Opposite: table of math layout handler objects, showing the number of arguments each object takes during processing and its purpose.

Object	Arguments	Purpose
accent	1	Displays accent over base character(s)
box	1	Gives properties to base
boxed formula	1	Displays borders and/or lines through base
delimiters	1	Encloses base in parens, brackets, braces, etc.
delimiters with separators	n	Encloses bases separated by separator character, such as a vertical bar
equation array	n	Displays set of horizontally aligned equations
fraction	2	Displays normal or small built-up fraction
function apply	2	Displays trigonometric and other functions with function name and base
left subsup	3	Prefixes a subscript and/or superscript to base
lower limit	2	Displays limit below base
matrix	n×m	Displays matrix with n columns and m rows
n-ary	3	Displays large n-ary operator with a base and optional upper and lower limits
operator character	1	Used internally to give proper spacing to operators
overbar	1	Displays bar over base (boxed formula special case)
phantom	1	Suppresses any combination of base ascent, descent, width, display, or transparency
radical	1, 2	Displays square and nth roots
slashed fraction	2	Displays slashed or built-up linear fraction
stack	2	Displays first argument over second (like fraction but without bar)
stretch stack	1	Displays stretchable character above or below base, or limit above or below stretchable base character
subscript	2	Displays subscript relative to base
subsup	3	Display subscript and superscript relative to base
superscript	2	Display superscript relative to base
underbar	1	Display bar under base (boxed formula special case)
upper limit	2	Display limit above base

An example: displaying $E = mc^2$

Murray Sargent

Technical text consists of normal text with interspersed math zones. These zones contain all the mathematics and may contain as little as a single mathematical variable, or one or more complete equations. The layout process begins when a layout client, *e.g.* Microsoft Word, requests the line layout handler (LLH) to display the next line of text. The LLH then calls the "fetch-run" callback for the line's first run of text. This callback distinguishes between text, math zones, and other in-line objects. When the LLH gets back the start of a math-zone object, it transfers control to the math layout handler (MLH). The MLH executes callbacks to the client to obtain math-zone properties, such as display versus in-line modes, zone ascent/descent, and a pointer to a client's bag of properties that will be passed to the client in most subsequent callbacks in the math zone. The MLH then calls the fetch-run callback for the first run of the math zone. The fetch-run callback distinguishes between mathematical text, mathematical objects and mathematical operators. The operators are treated as special single-character mathematical objects.

For example, consider Einstein's most famous equation, $E = mc^2$. The E is in its own text run, the equal sign is a mathematical operator object, the m is in its own text run, and the c^2 is a superscript object with text runs for arguments. The text runs result in various callbacks to obtain character properties, widths, and glyphs, as well as to display the glyphs or variants thereof once the whole line is laid out. All text is treated using glyphs and glyph-ink measurements of ascents and descents. The math italic letters E m and c are encoded as Unicode math alphabetics. The operator object for the equal sign results in callbacks to determine the operator's text characteristics and its default spacing class, in this case, relational. The superscript object results in callbacks to get text-run information for the base and superscript text, as well as to obtain the superscript vertical shift and the cut-in values for the upper-right corner of the c and lower-left corner of the ^{2}. These displacements are obtained from the math font handler, which is responsible for access to the math font's MATH table along with appropriate scaling. When the glyph for the superscript ^{2} is fetched, the math font handler is requested to return a script level-1 glyph variant with a relative size specified by the font (typically about 70% of the text size).

This example shows how even a short and relatively simple equation involves interplay between the client, the math layout handler, the math font handler, and the font itself. More complicated examples, such as those shown in the following specimen pages, involve math objects like brackets or integrals and need glyph assemblies and other information. In addition, larger equations may need to be wrapped to two or more lines, a process that involves further callbacks and information.

Mathematical sample settings

Sample settings

The following pages present specimens of the Cambria Math font used to typeset equations from a variety of mathematical and scientific disciplines. They are selected to illustrate many different aspects of the font and math layout behaviour discussed earlier in this book, as well as features that, due to space constraints, have not been covered in detail.

The creation of the sample settings is itself of some interest in the context of the development and testing of the the math font and layout handler. In order to test font behaviour and layout accuracy, Microsoft developed a test application with a large number of pre-loaded equations, as well as the ability to accept user input of additional test cases. This test tool was used to generate some of the sample settings on these specimen pages. Other samples are extracts from mathematical documents created using Microsoft's math typesetting solutions.

$$v = V\left[1 - \sum_{s=1}^{\infty} \frac{\pi J_1^2(b\beta_s)}{[J_1^2(b\beta_s) - J_0^2(a\beta_s)]} \{J_0(b\beta_s)Y_0(a\beta_s) - Y_0(b\beta_s)J_0(a\beta_s)\}\, exp(-\kappa\beta_s^2 t)\right]$$

Note that the equation above is the same as shown in the screen rendering image on page 38. The legibility of the latter can be compared to the high resolution print version shown here.

$$\begin{aligned}\int u dA &= \int_0^R r\,dr \int_0^{2\pi} d\theta \frac{\sigma}{2} z_0^2 \left(\left(1 - {r^2}/{R^2}\right) \omega \sin \omega t\right)^2 \\ &= \frac{\sigma}{2} 2\pi z_0^2 \omega^2 \sin^2 \omega t \int_0^R dr\, r \left(1 - {r^2}/{R^2}\right)^2 \\ &= \sigma\pi z_0^2 \omega^2 \sin^2 \omega t \frac{1}{2} \int_{r=0}^{r=R} d\left(r^2\right) \left(1 - {r^2}/{R^2}\right)^2 \\ &= \sigma\pi z_0^2 \omega^2 \sin^2 \omega t \frac{1}{2} R^2 \frac{1}{3} \left(1 - {r^2}/{R^2}\right)^3 (-1) \Big|_0^{r=R} \\ &= \frac{1}{6} \sigma R^2 \pi z_0^2 \omega^2 \sin^2 \omega t\end{aligned}$$

$$\begin{aligned}\frac{dT}{dt} &= \left(\frac{\partial T}{\partial r}\right)_\theta \frac{dr}{dt} + \left(\frac{\partial T}{\partial \theta}\right)_r \frac{d\theta}{dt}\\ &= \left(\frac{\partial T}{\partial r}\right)_\theta \left[\left(\frac{\partial r}{\partial x}\right)_y \frac{dx}{dt} + \left(\frac{\partial r}{\partial y}\right)_x \frac{dy}{dt}\right]\\ &= \left(-2T_1 \frac{r}{a^2}\right)\left[\frac{y}{\sqrt{x^2+y^2}} v_0\right]\\ &= -2T_1 \frac{\sqrt{c^2+v_0^2t^2}}{a^2} \cdot \frac{v_0^2 t}{\sqrt{c^2+v_0^2t^2}} = -\frac{2T_1 v_0^2 t}{a^2}\end{aligned}$$

$$\pi(n) = \sum_{m=2}^{n} \left[\left(\sum_{k=1}^{m-1} \left\lfloor \left({}^{m}\!/_{k}\right) / \left\lceil {}^{m}\!/_{k} \right\rceil \right\rfloor \right)^{-1}\right]$$

$$\boxed{\left\langle \phi_\lambda^{(i)} \middle| A_\kappa^{(j)} \middle| \psi_\mu^{(k)} \right\rangle = (i\lambda|j\kappa k\mu) \left\langle \phi^{(i)} \middle\| A^{(j)} \middle\| \psi^{(k)} \right\rangle}$$

$$A = A_{i_0} \xrightarrow{\phi_{i_0}} B_{i_0} = A_{i_1} \xrightarrow{\phi_{i_1}} \cdots \xrightarrow{\phi_{i_{n-1}}} B_{i_{n-1}} = A_{i_n} \xrightarrow{\phi_{i_n}} B_{i_n} = B$$

$$\sum_{j=1}^{n} \left(x_j^\gamma \Big/ \prod_{1\le k\le n, k\ne j} (x_j - x_k) \right) = \begin{cases} 0, & \text{if } 0 \le \gamma < n-1;\\ 1, & \text{if } \gamma = n-1;\\ \displaystyle\sum_{j=1}^{n} x_j, & \text{if } \gamma = n. \end{cases}$$

$$\begin{array}{ccccc} & & & \rightarrow & \searrow \\ & \nearrow & \rightarrow & {}^{15}\mathrm{N} + p^+ \rightarrow \alpha + {}^{12}\mathrm{C} & {}^{15}\mathrm{N} + p^+ \rightarrow {}^{16}\mathrm{O} + \gamma \\ & & & \downarrow & \downarrow \\ & {}^{15}\mathrm{O} + e^+ \rightarrow {}^{15}\mathrm{N} + \vec{\nu} & & {}^{12}\mathrm{C} + p^+ \rightarrow {}^{13}\mathrm{N} + \gamma & {}^{16}\mathrm{O}^+ \rightarrow {}^{17}\mathrm{F} + \gamma \\ & \uparrow & & \downarrow & \downarrow \\ & {}^{14}\mathrm{N} + p^+ \rightarrow {}^{15}\mathrm{O} + \gamma & & {}^{13}\mathrm{N} \rightarrow {}^{13}\mathrm{C} + e^+ + \nu & {}^{17}\mathrm{F} \rightarrow {}^{17}\mathrm{O} + e^+ + \nu \\ & & & \downarrow & \downarrow \\ & \nwarrow & \leftarrow & {}^{13}\mathrm{C} + p^+ \rightarrow {}^{14}\mathrm{N} + \gamma & {}^{17}\mathrm{O} + p^+ \rightarrow \alpha + {}^{14}\mathrm{N} \\ & & & \leftarrow & \swarrow \end{array}$$

$$\Theta = \left\{ \underline{\theta}: \theta_y \geq 0, \forall_y \in Y \text{ and } \sum_y \theta_y = 1; \theta_{ky} \geq 0, \bar{\theta}_{ky} \geq 0 \text{ and } \theta_{ky} + \bar{\theta}_{ky} = 1, \forall_y \in Y, \forall k \right\}$$

$$P(y \mid x; \underline{\theta}) = Z(x; \underline{\theta})^{-1} \cdot \theta_y \prod_{k=1}^{F} \left(\lambda_y \cdot \theta_{ky} + \bar{\lambda}_y \cdot \frac{1}{2} \right)^{f_k(x)} \left(\lambda_y \cdot \bar{\theta}_{ky} + \bar{\lambda}_y \cdot \frac{1}{2} \right)^{\overline{f_k(x)}}$$

$$\hat{\theta}_{ky} = N_{ky}^{-1} \cdot \theta_{ky} \cdot \left\{ 1 + \beta_{\underline{\theta}} \frac{\lambda_y}{\lambda_y \cdot \theta_{ky} + \bar{\lambda}_y \cdot \frac{1}{2}} \sum_{i=1}^{T} f_k(x_i) [\delta(y, y_i) - p(y \mid x_i; \underline{\theta})] \right\}$$

$$\int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin{(\alpha x)} dx = 2\pi\xi \left\{ \sum_{\xi[z]>0} Res \left[\frac{P(z)}{Q(z)} e^{i\alpha z} \right] \right\}$$

In general, we have the following formula.

The Binomial Theorem

If k is a positive integer, then

$$\begin{aligned}(a+b)^k = a^k + ka^{k-1}b + \frac{k(k-1)}{1 \cdot 2} a^{k-2}b^2 \\ + \frac{k(k-1)(k-2)}{1 \cdot 2 \cdot 3} a^{k-3}b^3 + \cdots \\ + \frac{k(k-1)\cdots(k-n+1)}{1 \cdot 2 \cdot 3 \cdot \cdots \cdot n} a^{k-n}b^n \\ + \cdots + kab^{k-1} + b^k\end{aligned}$$

EXAMPLE 13 Expand $(x-2^5$.

SOLUTION Using the Binomial Theorem with $a = x, b = -2, k = 5$, we have

$$\begin{aligned}(x-2)^5 &= x^5 + 5x^4(-2) + \frac{5 \cdot 4}{1 \cdot 2} x^3(-2)^2 + \frac{5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3} x^2(-2)^3 + 5x(-2)^4 + (-2)^5 \\ &= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32\end{aligned}$$

$$y_2(x) = y_1(x) \int^x \frac{\exp\left(-\int_a^{x_2} \sum_{i=-1}^{\infty} p_i x_1^i \, dx_1\right)}{x_2^{2\alpha}\left(\sum_{\lambda=0}^{\infty} a_\lambda x_2^\lambda\right)^2} dx_2,$$

where the solutions y_1 and y_2 have been normalized so that the Wronskian, $W(a) = 1$. Tackling the exponential factor first, we have

$$\int_a^{x_2} \sum_{i=-1}^{\infty} p_i x_1^i \, dx_1 = p_{-1} \ln x_2 + \sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1} + f(a).$$

Hence

$$\begin{aligned}
&\exp\left(-\int_a^{x_2} \sum_i p_i x_1^i \, dx_1\right) \\
&\quad = \exp[-f(a)] x_2^{-p_{-1}} \exp\left(-\sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1}\right) \\
&\quad = \exp[-f(a)] x_2^{-p_{-1}} \left[1 - \sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+1} + \frac{1}{2!}\left(\sum_{k=0}^{\infty} \frac{p_k}{k+1} x_2^{k+2}\right)^2 + \cdots\right]
\end{aligned}$$

This final series expansion of the exponential is certainly convergent if the original expansion of the coefficient $P(x)$ was convergent.

The denominator in Eq. 8.69 may be handled by writing

$$\begin{aligned}
\left[x_2^{2\alpha}\left(\sum_{\lambda=0}^{\infty} a_\lambda x_2^\lambda\right)\right]^{-1} &= x_2^{-2\alpha}\left(\sum_{\lambda=0}^{\infty} a_\lambda x_2^\lambda\right)^{-2} \\
&= x_2^{-2\alpha} \sum_{\lambda=0}^{\infty} b_\lambda x_2^\lambda.
\end{aligned}$$

Conclusion

The development of support for mathematical typesetting in Microsoft products has been a major undertaking, involving more than forty people—software developers, managers and testers from many different groups at Microsoft, external type designers and font technicians, experts in mathematics and typography—, whose collaboration is celebrated in this book. The goal of this undertaking is the same that has guided earlier collaborations between mathematicians and typographers: to express mathematical ideas and concepts in clear, succinct and beautiful visual form.

Mathematicians and scientists are 'symbol-hungry', and neither the current Unicode math character encoding nor Microsoft's implementation is exhaustive or closed to further extension. The work recorded in this book is a significant advance in many respects: in providing access to mathematical typesetting in the context of common word processing tools and a widely supported font format, in automating refinements to spacing and positioning that previously had to be made manually, and in addressing directly the appearance of mathematical text on screen. But this work is only the latest instance of an historic collaboration that will continue.

Despite the complexity of math layout requirements and the specialised features of math fonts and the tools for making them, type designers and typographers should not be hesitant in engaging in this ongoing collaboration. One of the things that became apparent in the course of this project was just how enthusiastic mathematicians and scientists are to see their work well-typeset and how appreciative they are of the efforts involved.

The material cost of producing new fonts for metal and even photo-typesetting effectively limited the range of typographic options available to mathematicians. It was always easier and cheaper to add new symbols to an existing typeface than to start afresh. With the advent of computer typography, the options expanded a little—notably with the collaboration of Hermann Zapf with Donald Knuth on the AMS Euler types—, but mathematical typesetting has yet to fully benefit from the digital revolution with extensive options in basic type styles. The Cambria Math font breaks new ground, not only via its technical features but also in demonstrating what is possible in terms of fresh approaches to the design of mathematical typesetting. We hope this project will inspire others.